Anyons, group theory and planar physics

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Abstract

Relativistic and nonrelativistic anyons are described in a unified formalism by means of the coadjoint orbits of the symmetry groups in the free case as well as when there is an interaction with a constant electromagnetic field. To deal with interactions we introduce the extended Poincaré and Galilei Maxwell groups.

1 Introduction

The first works, of a rather theoretical character, devoted to particles with an arbitrary spin and statistics in (2+1) dimensions go back to 1977 [1], but the real interest in physics behind the anyons started some years later when the fractional quantum Hall effect was explained just in terms of anyons [2].

In the last years some works analyzing the role of the Galilei and Poincaré groups in (2+1) dimensions in the theory of anyons have appeared in the literature [3]-[10]. Although the usual group theoretical considerations fit well when the anyons are free, several difficulties appear when the action of an external electromagnetic field is considered [11]. However, recently [12, 13] we have introduced two non-central extensions of the Poincaré and Galilei groups by homogeneous and constant electromagnetic fields, called Maxwell groups [14, 15] that seem to provide an appropriate group theoretical framework for anyons in the presence of constant fields. Our intention here is to adopt this viewpoint to present an unified approach for these kind of interacting systems in 2-space + 1-time dimensions which can also involve non-commuting coordinates. In this context we mention a recent work [16] where the authors also obtain, in a different way, the extended Galilei-Maxwell group (called by them "enlarged Galilei group").

The paper has been organized as follows. In the first two Sections we revise the Poincaré and Galilei groups to recover the main features of the free relativistic and nonrelativistic

anyons and fix the notation. As in all the cases presented along this paper, we have made a systematic use of the coadjoint orbit method that supply us with a canonical setup of classical systems bearing enough symmetry in terms of the corresponding symmetry group; in particular, for the Galilei group we have taken into account its double central extension. In the next two sections we deal with interacting anyons and constant electromagnetic fields. First, in Section 4 within a relativistic frame, while the nonrelativistic situation is considered in the following section, where we also discuss how to perform the nonrelativistic limit. As we mentioned before the groups involved in our analysis are certain noncentral extensions of the Galilei and Poincaré groups where the key point is that electromagnetic fields take part as dynamical objects. Some conclusions and comments on the main differences with other approaches end the paper.

We have also added two Appendices for the sake of completness. In a first Appendix we give a brief review about the symplectic structures associated to a Lie group. The second Appendix supply with a classification of orbits for the space $\mathcal{GM}^*(2+1)$ dual to the Lie algebra of the Galilei-Maxwell group.

2 Anyons and the Poincaré group

The Poincaré group $P(2+1)$, is a 6-dimensional (6-D) Lie group of transformations of the (2+1)–D Minkowski spacetime provided with the metric tensor $g_{ij} = \text{diag }(1, -1, -1)$. Two spatial cartesian axes will be denoted by X_1 and X_2.

Our (2+1)–D system may be seen as embedded in the (3+1)–D Minkowski spacetime equipped with the metric tensor $g_{ij} = \text{diag }(1, -1, -1, -1)$. Then, the third spatial axis perpendicular to the X_1X_2-plane will be denoted by X_3. We will make use of this embedding; for example a rotation on the plane X_1X_2 may be considered as the rotation around X_3. In that case vectors on the spatial plane are thought to be 3–D objects.

Each element of $P(2+1)$ is parametrized by a pair (a, Λ), where $a = (b, a_1, a_2)$ represents a time (b) and space (a_1, a_2) translation, and Λ a Lorentz transformation. The element Λ can be factorized as $\Lambda = \Lambda(\chi, \vec{n})\Lambda(R_\phi)$, with $\Lambda(\chi, \vec{n})$ being a boost of rapidity χ in the direction of the unit planar vector $\vec{n}$ and $\Lambda(R_\phi)$ denoting a rotation of angle ϕ around the axis X_3.

The Lie algebra $\mathcal{P}(2+1)$ of the Poincaré group $P(2+1)$ is spanned by the basis $\{P_0 \equiv H, P_1, P_2, K_1, K_2, J\}$, which are the infinitesimal generators of time and space translations, boosts transformations along axes X_1, X_2, and X_3-rotation, respectively. The nonvanishing commutators are

$$\begin{aligned} &[H, K_i] = -P_i, && [P_i, K_i] = -H, && [P_i, J] = -\varepsilon_{ij} P_j, \\ &[K_i, K_j] = -\varepsilon_{ij} J, && [K_i, J] = -\varepsilon_{ij} K_j, && i, j = 1, 2, \end{aligned} \tag{2.1}$$

where ε_{ij} denotes the 2–D completely skewsymmetric tensor.

2.1 Coadjoint orbits

Let $(h, p_1, p_2, k_1, k_2, j)$ be the coordinates of an arbitrary point of $\mathcal{P}^*(2+1)$, the dual space of $\mathcal{P}(2+1)$, in a basis dual to $\{H, P_1, P_2, K_1, K_2, J\}$. The coadjoint action of $P(2+1)$ on

$\mathcal{P}^*(2+1)$ is given by [17]

$$
\begin{aligned}
&g:(h,p_1,p_2,k_1,k_2,j)\rightarrow(h',p_1',p_2',k_1',k_2',j'),\qquad g=(b,a_1,a_2,\Lambda(\chi,\vec n),\,\Lambda(R_\phi)),\\
h' &= \cosh\chi\, h-\sinh\chi\,\vec n\cdot\vec p^{\,\phi},\\
\vec p\,' &= \vec p^{\,\phi}-\sinh\chi\, h\,\vec n+(\cosh\chi-1)(\vec n\cdot\vec p^{\,\phi})\,\vec n,\\
\vec k' &= \vec k^{\,\phi}+\sinh\chi\,\vec j\times\vec n^{\,\pi/2}-(\cosh\chi-1)(\vec n^{\,\pi/2}\cdot\vec k^{\,\phi})\,\vec n^{\,\pi/2}\\
&\quad+b\,[\vec p^{\,\phi}-\sinh\chi\, h\,\vec n+(\cosh\chi-1)(\vec n\cdot\vec p^{\,\phi})\,\vec n]\\
&\quad+\vec a\,[\cosh\chi\, h-\sinh\chi\,\vec n\cdot\vec p^{\,\phi}],\\
\vec j' &= \cosh\chi\,\vec j+\sinh\chi\,\vec n\times\vec k^{\,\phi}\\
&\quad+\vec a\times[\vec p^{\,\phi}-\sinh\chi\, h\,\vec n+(\cosh\chi-1)(\vec n\cdot\vec p^{\,\phi})\,\vec n],
\end{aligned}
\tag{2.2}
$$

where we have used the following notation:

$$
\vec n=(n_1,n_2,0),\quad \vec p=(p_1,p_2,0),\quad \vec k=(k_1,k_2,0),\quad \vec a=(a_1,a_2,0),\quad \vec j=(0,0,j),\tag{2.3}
$$

and $\vec x^{\,\phi}$ stands for the rotation of a vector $\vec x$ around the axis X_3 by an angle ϕ.

The invariants of the coadjoint action (2.2) are

$$
C_1=\rho^2{=}g^{\mu\nu}\rho_\mu\rho_\nu=h^2-p_1^2-p_2^2,\qquad C_2=h\,\vec j+\vec p\times\vec k,\tag{2.4}
$$

where $\rho=(h,p_1,p_2)$ and $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$.

The invariant C_2 is in fact a 3–D vector, but its only non-zero component is the third one, equal to $hj+p_1k_2-k_1p_2$. It is a lower dimensional version of the Pauli-Lubanski four-vector. Recall that in $(3+1)$–D Minkowski spacetime the Pauli-Lubanski vector w takes the form

$$
w{=}(w^0,\mathbf{w})=(\mathbf{j}\cdot\mathbf{p},h\,\mathbf{j}+\mathbf{p}\times\mathbf{k}),\tag{2.5}
$$

where now the involved vectors are generic 3–D. The scalar w^2 is invariant under the (3+1)–Poincaré group action. In (2+1) dimensions the Pauli-Lubanski vector reduces to the expression (2.4) of C_2.

The classification of the coadjoint orbits was published in [17, 18]. There are orbits of dimension 4, 2 and 0 (points). The 4–D orbits are divided in three classes:

- $C_1>0$ relativistic particles of a mass $\sqrt{C_1}$,
- $C_1<0$ tachyons,
- $C_1=0,\ \vec p\neq 0$ massless particles.

We will consider the strata of orbits with $C_1>0$. Rewriting $C_1=m^2$, where m is the rest mass of the particle, the invariant C_1 leads to the equation of a hyperboloid of two sheets $h^2-p_1^2-p_2^2-m^2=0$. We will restrict to that one of positive energy $h>0$, denoted H_m^+, as usual. For the second invariant, we rewrite $C_2=m\,\vec s$, where

$$
\vec s=\frac{h\,\vec j+\vec p\times\vec k}{\sqrt{h^2-p_1^2-p_2^2}}\ .\tag{2.6}
$$

The nontrivial component of $\vec s$ is the spin s of the system.

2.2 Symplectic structure

The two independent invariants m and s fix, in the way presented above, a coadjoint orbit $\mathcal{O}^+_{m,s}$ that constitutes a 4–D differentiable submanifold of $\mathcal{P}^*(2{+}1)$. Moreover, we can cover $\mathcal{O}^+_{m,s}$ with one chart $(\mathcal{O}^+_{m,s}, \varphi)$ using as coordinates (p_1, p_2, k_1, k_2). Indeed, the Jacobian of the transformation

$$(h, p_1, p_2, k_1, k_2, j) \to (C_1, C_2, p_1, p_2, k_1, k_2)$$

being $(2h^2)^{-1}$, it is always positive on the sheet H^+_m. On the orbits $\mathcal{O}^+_{m,s}$ there is a natural Poisson structure (A.5) —see Appendix A— given by

$$\Lambda = h\, \frac{\partial}{\partial k_i} \wedge \frac{\partial}{\partial p_i} - j\, \frac{\partial}{\partial k_1} \wedge \frac{\partial}{\partial k_2}\ . \tag{2.7}$$

The symplectic form related with the tensor Λ is

$$\omega = -\frac{1}{h}\, dk_i \wedge dp_i - \frac{j}{h^2}\, dk_1 \wedge dk_2\, , \tag{2.8}$$

where, according to (2.4)

$$h = +\sqrt{\vec{p}^{\,2} + m^2}, \qquad j = \frac{ms - p_1 k_2 + p_2 k_1}{\sqrt{\vec{p}^{\,2} + m^2}}\ . \tag{2.9}$$

The coordinates, $p_i,\ k_i,\ i = 1, 2$ are not canonical since their Poisson brackets are

$$\{k_1, k_2\} = -j, \qquad \{p_1, p_2\} = 0, \qquad \{k_i, p_j\} = h\, \delta_{ij}. \tag{2.10}$$

The equations of the time evolution obtained from the law of motion (A.6) with the Hamiltonian (2.9) are

$$\dot{p}_i = 0, \qquad \dot{k}_i = p_i, \quad i = 1, 2. \tag{2.11}$$

They look like the equations of motion of a nonrelativistic free particle.

We find a set of canonical coordinates $\{\vec{p}, \vec{q}\}$, where

$$\vec{q} = \frac{\vec{k}}{h} - \frac{\vec{p} \times \vec{s}}{h(m+h)} \tag{2.12}$$

and the expression for the angular momentum becomes

$$j = \frac{ms}{h} + \vec{q} \times \vec{p} + \frac{ms}{h(m+h)}\vec{p}^{\,2}. \tag{2.13}$$

Now, if we identify q_i as 'position coordinates', the equations of motion are the well-known relations

$$\dot{p}_i = 0, \qquad \dot{q}_i = \frac{p_i}{h}, \qquad i = 1, 2. \tag{2.14}$$

A detailed analysis of the different coordinate systems for anyons can be found in Ref. [19].

2.3 Irreducible unitary representations

In quantum mechanics the coadjoint orbits of a Lie group allow us to define the irreducible unitary representations (IUR) associated to quantum elementary physical systems having such a symmetry group. Thus, the IUR's of $P(2{+}1)$ associated to the stratum of orbits $\mathcal{O}^+_{m,s}$ are

$$[U_{m,s}(a,\Lambda)]\psi(p) = e^{ip\cdot a} e^{is\,\theta(p,\Lambda)}\psi(\Lambda^{-1}p), \tag{2.15}$$

where s is the quantum number labelling a representation of $SO(2)$ and $\theta(p,\Lambda)$ is the Wigner angle, which is determined by the little group of a point of the orbit. More explicitly, choosing the point $p_m = (m,0,0)$, whose isotropy group is $SO(2)$, and the boost elements $\Lambda_{p\to p'} \in SO(2,1)$ transforming the point p into p', then

$$\theta(p,\Lambda) = \Lambda^{-1}_{p_m\to\Lambda(p)}\,\Lambda\,\Lambda_{p_m\to p}. \tag{2.16}$$

The functions $\psi(p)$ belong to the Hilbert space $\mathcal{H} = \mathcal{L}^2(H^+_m, d\mu(p))$, being $d\mu(p)$ the $SO(2,1)$–invariant measure in H^+_m.

The differential realization of the generators for this representation is

$$\hat{H} = h, \quad \hat{P}_i = p_i, \quad \hat{K}_j = i(h\partial_{p_j} + p_j\partial_h) + \frac{\varepsilon^{jk}p_k}{2m}s, \quad \hat{J} = i(p_2\partial_{p_1} - p_1\partial_{p_2}) + \frac{h}{m}s. \tag{2.17}$$

The two Casimirs corresponding to the invariants (2.4), $C_1 = \hat{P}^2$ and $C_2 = \hat{H}\cdot\hat{J} + \vec{\hat{P}}\times\vec{\hat{K}}$, give the following equations

$$(p^2 - m^2)\,\psi(p) = 0, \qquad (h\,\hat{J} + \vec{\hat{P}}\times\vec{\hat{K}} - ms)\,\psi(p) = 0. \tag{2.18}$$

The first one corresponds to the mass shell condition which gives rise to the Klein-Gordon equation. The second one is the Pauli–Lubanski equation describing the spin of the particle. In two dimensions the unitarity of the realizations does not impose restrictions on the values of s, thus allowing for the existence of anyons. In this way we easily recover results of Ref. [4].

3 Nonrelativistic anyons and the Galilei group

In the nonrelativistic case we have to deal with the Galilei group $G(2{+}1)$ in (2+1)–D, which can be seen as a contraction of the Poincaré group $P(2{+}1)$. The commutation rules of its Lie algebra $\mathcal{G}(2{+}1)$ are those of Poincaré (2.1) except that now

$$[K_1,K_2] = 0, \qquad [K_i,P_j] = 0, \qquad i,j = 1,2. \tag{3.1}$$

The algebra $\mathcal{G}(2{+}1)$ admits a 2–D central extension $\bar{\mathcal{G}}(2{+}1)$ characterized by the new commutators [20, 21]

$$[P_i,K_j] = -\delta_{ij}M, \qquad [K_1,K_2] = \mathcal{K}, \tag{3.2}$$

where M and $\mathcal{K}$ are central generators, i.e., $[M,\cdot] = [\mathcal{K},\cdot] = 0$ for any element of $\mathcal{G}(2{+}1)$.

Both extensions can also be obtained by a contraction from the Poincaré group [21, 22]. It is enough to consider the direct product $\tilde{P}(2{+}1) = \mathbb{R}^2 \otimes P(2{+}1)$. Obviously, at the level of the

Lie algebra we have $\tilde{\mathcal{P}}(2{+}1) = \mathbb{R}^2 \oplus \mathcal{P}(2{+}1)$. Hence, a basis is constituted by the generators of $\mathbb{R}^2$ $(M, \mathcal{K})$ plus the known generators of the Poincaré algebra $(H, P_1, P_2, K_1, K_2, J)$. Let us consider a new basis given by

$$M' = M, \quad \mathcal{K}' = \mathcal{K}, \quad H' = H - M, \quad P_i' = P_i, \quad K_i' = K_i, \quad J' = J + \mathcal{K}, \qquad i = 1,2. \tag{3.3}$$

The nonvanishing commutators of $\tilde{\mathcal{P}}(2{+}1)$ in this new basis are

$$\begin{array}{lll} [H', K_i'] = -P_i', & [P_i', K_i'] = -H' - M', & [P_i', J'] = -\varepsilon_{ij} P_j', \\ [K_i', K_j'] = -\varepsilon_{ij}(J' - \mathcal{K}'), & [K_i', J'] = -\varepsilon_{ij} K_j', & i,j = 1,2. \end{array} \tag{3.4}$$

Now in order to perform the contraction we define an appropriate rescaled basis

$$M'' = \epsilon^2 M', \quad \mathcal{K}'' = \epsilon^2 \mathcal{K}', \quad H'' = H', \quad P_i'' = \epsilon P_i', \quad K_i'' = \epsilon K_i', \quad J'' = J', \tag{3.5}$$

where ϵ is a fixed real positive number. The nonvanishing Lie commutators are now

$$\begin{array}{lll} [H'', K_i''] = -P_i'', & [P_i', K_i'] = -\epsilon^2 (H'' + \frac{1}{\epsilon^2} M''), & [P_i'', J''] = -\varepsilon_{ij} P_j'', \\ [K_i'', K_j''] = -\varepsilon_{ij} \epsilon^2 (J'' - \frac{1}{\epsilon^2} \mathcal{K}''), & [K_i'', J''] = -\varepsilon_{ij} K_j'', & i,j = 1,2. \end{array}$$

In the limit $\epsilon \to 0$ we recover the Lie commutators of the extended algebra $\bar{\mathcal{G}}(2{+}1)$.

To give a physical interpretation of the contraction procedure we identify the contraction parameter ϵ with the inverse of the light speed ($\epsilon = 1/c$). From a cohomological point of view a change of the basis defined by relations (3.3)-(3.5) corresponds to introducing a trivial two-cocycle on the Poincaré group. After the contraction $\epsilon \to 0$ this trivial two-cocyle becomes a non-trivial one of the Galilei group [23].

3.1 Coadjoint orbits

By $\bar{\mathcal{G}}^*(2{+}1)$ we will denote the space dual to the algebra $\bar{\mathcal{G}}(2{+}1)$. Each vector belonging to $\bar{\mathcal{G}}^*(2{+}1)$ is characterized by eight components $(m, \kappa, h, p_1, p_2, k_1, k_2, j)$ in the basis dual to $(M, \mathcal{K}, H, P_1, P_2, K_1, K_2, J)$ of $\bar{\mathcal{G}}(2{+}1)$.

Let us denote by $g = (\theta, \eta, b, \vec{a}, \vec{v}, R_\phi)$ the elements of $\bar{G}(2{+}1)$, with a convention similar to that of Poincaré $\mathcal{P}(2{+}1)$ except that, $\vec{v}$ stands for the Galilean boosts, and (θ, η) parametrize the group elements generated by $(M, \mathcal{K})$. The coadjoint action of $g \in \tilde{G}(2{+}1)$ on the dual space $\bar{\mathcal{G}}^*(2{+}1)$ is given by [21]

$$\begin{aligned} m' &= m, \\ \kappa' &= \kappa, \\ h' &= h - \vec{v} \cdot \vec{p}^{\,\phi} + \tfrac{1}{2} m \vec{v}^{\,2}, \\ \vec{p}^{\,\prime} &= \vec{p}^{\,\phi} - m \vec{v}, \\ \vec{k}^{\,\prime} &= \vec{k}^{\,\phi} + b\, \vec{p}^{\,\phi} + m\,(\vec{a} - b\,\vec{v}) + \vec{v} \times \vec{\kappa}, \\ \vec{j}^{\,\prime} &= \vec{j} + \vec{a} \times \vec{p}^{\,\phi} + \vec{v} \times \vec{k}^{\,\phi} - \tfrac{1}{2}\,\kappa\,\vec{v}^{\,2} - m\,\vec{a} \times \vec{v}, \end{aligned} \tag{3.6}$$

where we have also used the notation

$$\vec{p} = (p_1, p_2, 0), \quad \vec{v} = (v_1, v_2, 0), \quad \vec{k} = (k_1, k_2, 0), \quad \vec{a} = (a_1, a_2, 0), \quad \vec{\kappa} = (0, 0, \kappa), \quad \vec{j} = (0, 0, j).$$

The invariants of the coadjoint action (3.6), besides m and κ, are

$$C_1 = \vec{p}^{\,2} - 2\, m\, h, \qquad C_2 = m\, \vec{j} - \vec{\kappa}\, h + \vec{p} \times \vec{k}. \tag{3.7}$$

Note that the first one can be written as $U = -C_1/2m$ and is interpreted as the internal energy of the physical system. As in the relativistic case we denote $C_2 = m\vec{s}$, but now

$$\vec{s} = \vec{j} - \frac{\vec{\kappa} h}{m} + \frac{\vec{p} \times \vec{k}}{m}. \tag{3.8}$$

It is easy to derive the expressions (3.7) from their relativistic counterparts (2.4) following the contraction procedure outlined above (see also Ref. [22]). Obviously, expression (3.8) is the nonrelativistic limit of (2.6).

The classification of the coadjoint orbits (4–D, 2–D and 0–D) was presented in [21]. The relevant 4–D orbits characterized by the values $\{m \neq 0, \kappa, U, \ s\}$ are denoted by $\mathcal{O}^{\kappa,U}_{m,s}$. In the following we will assume $\kappa \neq 0$, since the results for $\kappa = 0$ can be obtained directly.

3.2 Symplectic structure

A set of coordinates adapted to the orbit $\mathcal{O}^{\kappa,U}_{m,s}$ are $(m, \kappa, U, s, p_1, p_2, x_1, x_2)$, where $x_i = k_i/m$. Since the transformation

$$(m, \kappa, h, p_1, p_2, k_1, k_2, j) \longrightarrow (m, \kappa, U, s, p_1, p_2, x_1 = k_1/m, x_2 = k_2/m)$$

has a nonzero Jacobian (as long as $m \neq 0$!), we can cover the whole orbit $\mathcal{O}^{\kappa,U}_{m,s}$ with one chart $(\mathcal{O}^{\kappa,U}_{m,s}, \varphi)$ using coordinates (p_1, p_2, x_1, x_2). The induced Poisson tensor Λ on the orbit $\mathcal{O}^{\kappa,U}_{m,s}$ takes the form

$$\Lambda = \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial p_i} + \frac{\kappa}{m^2} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \qquad i = 1, 2. \tag{3.9}$$

The inverse of Λ gives the symplectic form ω on $\mathcal{O}^{\kappa,U}_{m,s}$,

$$\omega = dx_i \wedge dp_i + \frac{\kappa}{m^2}\, dx_1 \wedge dx_2, \qquad i = 1, 2. \tag{3.10}$$

The coordinates $(\vec{p}, \vec{x})$ are not canonical since their Poisson brackets are

$$\{x_1, x_2\} = \frac{\kappa}{m^2}, \qquad \{p_1, p_2\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}. \tag{3.11}$$

Nevertheless, the Hamiltonian

$$h = \frac{\vec{p}^2}{2m} + U \tag{3.12}$$

has the usual form of a free nonrelativistic particle, leading to the motion equations

$$\dot{\vec{p}} = 0, \qquad \dot{\vec{x}} = \frac{\vec{p}}{m}. \tag{3.13}$$

We see that the dynamics of the particle is independent of the parameter κ because the Hamilton function (3.12) does not contain any function of $\vec{x}$. Equations (3.13) are consistent with (3.6) if we assume that the parameter b represents the time and the coadjoint action gives the relation between the coordinates in two different inertial frames. Indeed, let the laboratory frame Σ' and the (instantaneous) rest frame Σ of the particle be related by the element $g = (t, \vec{a}, -\vec{v}, 0) \in G(2{+}1)$. Then, equations (3.6) can be read as

$$\vec{p}_{\Sigma'} = m\vec{v}, \qquad \vec{x}_{\Sigma'} = t\,\frac{\vec{p}_{\Sigma'}}{m} - \frac{\vec{p}_{\Sigma'}}{m^2} \times \vec{\kappa} + \vec{a}. \tag{3.14}$$

The angular momentum is given by

$$j = \vec{x} \times \vec{p} + \frac{\vec{\kappa}}{m}\left(\frac{\vec{p}^{\,2}}{2m} + U\right) + s \tag{3.15}$$

which is the nonrelativistic limit of (2.13).

A set of canonical coordinates $(\vec{p}, \vec{q})$ can be straightforwardly obtained from $(\vec{p}, \vec{k})$ by

$$q_i = x_i + \frac{\varepsilon_{ij}\kappa p_j}{2m^2}. \tag{3.16}$$

It is worth to consider (3.16) as the nonrelativistic limit of the corresponding ones (2.12) for Poincaré. The coadjoint action of the Galilei group $G(2{+}1)$ in these coordinates is expressed by formulas

$$\begin{aligned} \vec{p}\,' &= \vec{p}^{\,\phi} - m\vec{v}, \\ \vec{q}\,' &= \vec{q}^{\,\phi} + \tfrac{b}{m}\,\vec{p}^{\,\phi} + (\vec{a} - b\,\vec{v}) + \tfrac{1}{2m}\vec{v} \times \vec{\kappa}. \end{aligned} \tag{3.17}$$

Although κ does not affect the dynamics, it gives the contribution $\frac{1}{2m}\vec{v} \times \vec{\kappa}$ to $\vec{q}$. The time evolution of canonical variables is given by (cf. (3.13))

$$\dot{p}_i = 0, \qquad \dot{q}_i = p_i, \qquad i = 1, 2. \tag{3.18}$$

Note that in the coordinates $(\vec{p}, \vec{q})$ the angular momentum takes the form

$$\vec{j} = \vec{q} \times \vec{p} + \frac{\vec{\kappa}\,U}{m} + s, \tag{3.19}$$

where κ gives rise to an extra term. Similarly to (3.14) equations (3.17) give now

$$\vec{p} = m\vec{v}, \qquad \vec{q} = t\,\frac{\vec{p}}{m} - \frac{\vec{p}}{2m^2} \times \vec{\kappa} + a. \tag{3.20}$$

The κ–term is the only one that remains without a clear physical interpretation in the free case [21, 24].

3.3 Irreducible unitary representations

The IUR of $G(2{+}1)$ associated to this stratum of orbits are [21]

$$[U^{\kappa,U}_{m,s}(g)]\psi(\vec{p}) = e^{i((\frac{1}{2m}\vec{p}^{\,2}+U)b - \vec{p}\cdot\vec{a})} e^{i\kappa(\frac{1}{2m}\vec{v}\times\vec{p})} e^{i(s+\kappa\frac{U}{m})\phi}\psi(R^{-1}(\phi)(\vec{p} - m\vec{v})), \tag{3.21}$$

where $R(\phi)$ is a rotation of angle ϕ. The carrier space of the representation is the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$. Note that the differential realization of the generators in this representation is

$$\hat{P}_j = p_j, \quad \hat{K}_j = im\partial_{p_j} - \frac{\kappa}{2m}\varepsilon^{jk}p_k, \quad \hat{H} = \frac{1}{2m}\vec{p}^{\,2} + U, \quad \hat{J} = i(p_2\partial_{p_1} - p_1\partial_{p_2}) + s + \kappa U. \tag{3.22}$$

After a global change of phase, $U'(g) = \lambda(g)U(g)$, the IUR $U^{\kappa,U}_{m,s}$ is shown to be equivalent to $U^{\kappa,0}_{m,0}$.

It is worth mentioning that we can consider massless particles in the Galilean framework. In this case their 4-D coadjoint orbits are characterized by $(m = 0, \kappa \neq 0)$ and the invariants $C_1 = \vec{p}^{\,2} \neq 0$ and $C_2 = \vec{p} \times \vec{k} - \kappa h$. If we compare C_2 with the relativistic Pauli-Lubanski operator for $m = 0$ in (2.18), $(h\ \hat{J} + \hat{\vec{P}} \times \hat{\vec{K}} - C_2)\ \psi(p) = 0$, we see that C_2 gives the helicity of our system, while J is now replaced by κ. We remark that this case is not equivalent to that of the orbit $(m = 0, \kappa = 0)$, and that in the present context the Hamiltonian becomes linear in $\vec{p}$. However, one must be careful about the interpretation since the coordinates $(\vec{k}, \vec{p})$ are not canonical (see also [4]).

The IUR associated to the null–mass orbits are

$$[U^{\kappa}_{C_1,C_2}(g)]\psi(\vec{w},\theta) = e^{i\kappa(\frac{1}{2}\vec{w}\times\vec{y})}e^{i\vec{p}\cdot(\vec{a}-b\vec{y})}e^{-i\frac{C_2}{2\kappa}b}\psi(\vec{y}-\vec{w},\theta-\phi), \tag{3.23}$$

where $\vec{p} = \sqrt{C_1}(\cos\theta, \sin\theta)$ and $\psi \in \mathcal{L}^2(\mathbb{R}^2 \times S^1)$.

4 Relativistic anyons in an external electromagnetic field

Once revisited the description of free anyons, in the next sections we will analyze the more interesting case when charged particles move in a constant electromagnetic field. The presence of external forces modifies the symmetry group of the system. This is the reason why instead of the Poincaré $P(2{+}1)$ and Galilei $G(2{+}1)$ groups, we will consider the so called Poincaré–Maxwell $PM(2{+}1)$ and Galilei–Maxwell $GM(2{+}1)$ groups [14, 15].

Let us start with the relativistic case. The Poincaré–Maxwell group $PM(2{+}1)$ is a 9–D Lie group with six infinitesimal generators $\{H, P_1, P_2, K_1, K_2, J\}$, corresponding to $P(2{+}1)$, plus three new elements $\{B, E_1, E_2\}$ related to the electromagnetic field [14]. It can be considered as a 3–D noncentral extension of the Poincaré group. The nonvanishing commutators for its Lie algebra, $\mathcal{PM}(2{+}1)$, are:

$$\begin{aligned}
&[B, K_i] = \varepsilon_{ij}E_j, && [E_i, K_j] = -\varepsilon_{ij}B, && [E_i, J] = -\varepsilon_{ij}E_j,\\
&[H, P_i] = E_i, && [H, K_i] = -P_i, && [P_i, P_j] = -\varepsilon_{ij}B,\\
&[P_i, K_j] = -\delta_{ij}H, && [P_i, J] = -\varepsilon_{ij}P_j, && [K_i, K_j] = -\varepsilon_{ij}J,\\
& && [K_i, J] = -\varepsilon_{ij}K_j, && \qquad i,j = 1,2.
\end{aligned} \tag{4.1}$$

4.1 Coadjoint orbits

We will denote by $(\beta, \epsilon_1, \epsilon_2, h, p_1, p_2, k_1, k_2, j)$ the coordinates fixing a point on $\mathcal{PM}^*(2{+}1)$ by means of the dual basis of $(B, E_1, E_2, H, P_1, P_2, K_1, K_2, J)$. The general formula expressing the coadjoint action of a group element $g = (c, \vec{d}, b, \vec{a}, \Lambda(\chi, \vec{n})\Lambda(R_\phi)) \in PM(2{+}1)$ on

$\mathcal{PM}^*(2{+}1)$ is given by

$$
\begin{aligned}
\vec{\beta}' &= (\cosh\chi)\vec{\beta} + (\sinh\chi)\,(\vec{n}\cdot\vec{\epsilon}^{\,\phi})\,\vec{n}^{\,\pi/2},\\
\vec{\epsilon}' &= \vec{\epsilon} - (\sinh\chi)(\vec{\beta}\times\vec{n}^{\,\pi/2}) + (\cosh\chi-1)(\vec{n}^{\,\pi/2}\cdot\vec{\epsilon}^{\,\phi})\,\vec{n}^{\,\pi/2},\\
h' &= (\cosh\chi)h - (\sinh\chi)(\vec{n}\cdot\vec{p}^{\,\phi}) - \vec{a}\cdot\vec{\epsilon}\,',\\
\vec{p}\,' &= \vec{p}^{\,\phi} - (\sinh\chi)h\,\vec{n} + (\cosh\chi-1)(\vec{n}\cdot\vec{p}^{\,\phi})\,\vec{n} + b\vec{\epsilon}\,' - \vec{\beta}'\times\vec{a},\\
\vec{k}\,' &= \vec{k}^{\phi} - (\sinh\chi)\vec{j}\times\vec{n}^{\,\pi/2} + (\cosh\chi-1)(\vec{n}^{\,\pi/2}\times\vec{k}^{\,\phi})\,\vec{n}^{\,\pi/2}\\
&\qquad + b\vec{p}\,' - \tfrac{1}{2}b^2\vec{\epsilon}\,' + h'\vec{a} + \tfrac{1}{2}\vec{a}^2\vec{\epsilon}\,' - \vec{\beta}'\times\vec{d} - \vec{c}\times\vec{\epsilon}\,',\\
\vec{j}\,' &= (\cosh\chi)\vec{j} + (\sinh\chi)\,(\vec{n}\times\vec{k}^{\phi}) + \vec{a}\times\vec{p}\,' + \tfrac{1}{2}\vec{a}^2\vec{\beta}' + \vec{d}\times\vec{\epsilon}\,',
\end{aligned}
\tag{4.2}
$$

where the notation used is the same as in (2.3) with the additional vectors

$$\vec{\beta} = (0,0,\beta), \qquad \vec{\epsilon} = (\epsilon_1,\epsilon_2,0), \qquad \vec{c} = (0,0,c), \qquad \vec{d} = (d_1,d_2,0).$$

The parameters c, d_1 and d_2 describe the group elements generated by B, E_1 and E_2, respectively. Notice that (h, p_1, p_2) represents the 3-D energy-momentum vector covariant under (2+1) Lorentz transformations.

The invariants under the coadjoint action (4.2) are

$$
\begin{aligned}
C_0 &= \vec{\epsilon}^{\,2} - \vec{\beta}^{\,2},\\
C_1 &= h^2 - \vec{p}^{\,2} - 2(\vec{k}\cdot\vec{\epsilon} - \vec{j}\cdot\vec{\beta}\,),\\
C_2 &= h\,\vec{\beta} + \vec{p}\times\vec{\epsilon}.
\end{aligned}
\tag{4.3}
$$

The first one is, of course, the invariant of the electromagnetic field under Lorentz transformations, $C_0 = -F_{\mu\nu}F^{\mu\nu}$, where $F_{0i} = \epsilon_i$ and $F_{12} = \beta$. If $C_0 > 0$, $C_0 < 0$ or $C_0 = 0$ we say that the electromagnetic field (or the orbit) is of electric, magnetic or perpendicular type, respectively.

The second invariant C_1 describes the interaction: it includes the electric coupling term $\vec{k}\cdot\vec{\epsilon}$, and the coupling of angular momentum and magnetic field, $\vec{j}\cdot\vec{\beta}$. The last invariant C_2 from (4.3) admits a covariant expression $C_2 = -\varepsilon^{\mu\nu\sigma}p_\mu F_{\nu\sigma}$. It has not an immediate interpretation, but its appearance is a consequence of the symmetries of the system.

Since we have three independent invariants the maximal dimension of the coadjoint orbits is 6. In this work we will be concerned only with this kind of orbits, henceforth denoted $\mathcal{O}^{C_0}_{C_1\,C_2}$.

4.2 Symplectic structure

A suitable chart of coordinates for the points of the 9-D differentiable manifold $\mathcal{PM}^*(2{+}1)$ is given by $(C_0, C_1, C_2, \epsilon_1, \epsilon_2, p_1, p_2, k_1, k_2)$. Each 6–D orbit $\mathcal{O}^{C_0}_{C_1\,C_2}$ can be covered with just one chart $(\mathcal{O}^{C_0}_{C_1\,C_2}, \varphi)$ with coordinates $(\epsilon_1,\, \epsilon_2,\, p_1,\, p_2,\, k_1,\, k_2)$. Indeed, from (4.3) we find that the Jacobian of the transformation

$$(\beta,\, h,\, j,\, \epsilon_1,\, \epsilon_2,\, p_1,\, p_2,\, k_1,\, k_2) \longrightarrow (C_0,\, C_1,\, C_2,\, \epsilon_1,\, \epsilon_2,\, p_1,\, p_2,\, k_1, k_2)$$

equals $4\beta^3$. A singularity appears if $\beta = 0$, but we shall deal here only with orbits of magnetic type ($\beta \neq 0$).

In the chart $(\mathcal{O}^{C_0}_{C_1\,C_2}, \varphi)$ the Poisson tensor Λ takes the form

$$\Lambda = -\beta\frac{\partial}{\partial\epsilon_1}\wedge\frac{\partial}{\partial k_2}+\beta\frac{\partial}{\partial\epsilon_2}\wedge\frac{\partial}{\partial k_1}-\beta\frac{\partial}{\partial p_1}\wedge\frac{\partial}{\partial p_2}-h\frac{\partial}{\partial p_1}\wedge\frac{\partial}{\partial k_1}-h\frac{\partial}{\partial p_2}\wedge\frac{\partial}{\partial k_2}-j\frac{\partial}{\partial k_1}\wedge\frac{\partial}{\partial k_2}, \quad (4.4)$$

where β, h, j are functions of the coordinates obtained from relations (4.3). To make easy further comparisons with the nonrelativistic case we will represent it as a matrix (using the previous order of the coordinates)

$$\Lambda^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta & -h & 0 \\ 0 & 0 & \beta & 0 & 0 & -h \\ 0 & -\beta & h & 0 & 0 & -j \\ \beta & 0 & 0 & h & j & 0 \end{pmatrix} \quad (4.5)$$

whose determinant is β^6. The symplectic two-form is

$$\omega = \frac{h^2 - j\beta}{\beta^3}d\epsilon_1\wedge d\epsilon_2+\frac{h}{\beta^2}d\epsilon_1\wedge dp_1+\frac{1}{\beta}d\epsilon_1\wedge dk_2\frac{h}{\beta^2}d\epsilon_2\wedge dp_2-\frac{1}{\beta}d\epsilon_2\wedge dk_1+\frac{1}{\beta}dp_1\wedge dp_2. \quad (4.6)$$

Such a symplectic form is not canonical: the nonvanishing Poisson brackets are

$$\{\epsilon_i, k_l\} = -\varepsilon_{il}\,\beta, \quad \{p_i, p_l\} = \varepsilon_{il}\,\beta, \quad \{p_i, k_l\} = -\delta_{il}h, \quad \{k_i, k_l\} = -\varepsilon_{il}\,j, \qquad i, l = 1, 2. \quad (4.7)$$

In order to find the equations of motion we need to know the Hamiltonian of the system. From the invariant C_2 (see (4.3)) we get

$$h = \frac{C_2}{\beta} + \frac{\vec{\epsilon}\times\vec{p}}{\beta}, \quad (4.8)$$

where from (4.3) we can write $\beta = \sqrt{\vec{\epsilon}^{\,2} - C_0}$. Thus, we obtain

$$\dot{\epsilon_i} = 0, \qquad \dot{p_i} = -\epsilon_i, \qquad \dot{k_i} = p_i, \qquad i = 1, 2. \quad (4.9)$$

Equations (4.9) are extremely simple. The first one says that the fields are constant. The others look like the equations of motion for a nonrelativistic particle with a unit negative charge inside an electric field. Comparing the time evolution from these formulae with the transformation rules (4.2) we can identify the time t with the parameter b. This is natural because b is associated to the Hamiltonian generator H.

Let us remark that the above relations are quite different from the usual ones derived from the standard Hamiltonian formalism. In that approach the Hamilton function is

$$H = \sqrt{m^2 + (\vec{\pi} + \frac{1}{2}\vec{\beta}\times\vec{r})^2} - \vec{\epsilon}\cdot\vec{r}, \quad (4.10)$$

where m denotes the mass of the particle and $\vec{r}$ its vector of position, being $(\vec{\pi}, \vec{r})$ canonically conjugated variables. Then, the equations of motion are

$$\dot{\vec{\pi}} = \vec{\epsilon} - \frac{1}{2}\dot{\vec{r}} \times \vec{\beta}, \qquad \dot{\vec{r}} = \frac{1}{\mathcal{E}}(\vec{\pi} + \frac{1}{2}\vec{\beta} \times \vec{r}\,), \tag{4.11}$$

with $\mathcal{E} = H + \vec{\epsilon} \cdot \vec{r}$ the energy of the system. But, the "group" coordinates $\epsilon_1, \epsilon_2, p_1, p_2, k_1, k_2$ have not a simple interpretation in terms of $\vec{\pi}$ and $\vec{r}$ (4.11). In fact, there is not a punctual transformation relating both pictures. We shall comment on this problem more carefully in the last section.

5 Nonrelativistic anyons in an external electromagnetic field

The magnetic limit [25] of the Poincaré-Maxwell group, that we call Galilei–Maxwell group [15], is the most suitable to describe non-relativistic anyons in the presence of external covariant fields.

The Galilei–Maxwell group is a 10–D Lie group, whose infinitesimal generators are those of $G(2{+}1)$, $\{H,\, P_1,\, P_2,\, K_1,\, K_2,\, J\}$, together with $\{E_1,\, E_2,\, B,\, M\}$. However, here, we will take into account also the central extension characterized by the nonvanishing commutator $[K_1, K_2] = \mathcal{K}$, leading to a group denoted simply $GM(2{+}1)$. The nonvanishing commutators of its Lie algebra, $\mathcal{GM}(2{+}1)$, are

$$\begin{array}{lll} & [E_i, K_j] = -\varepsilon_{ij} B & [E_i, J] = -\varepsilon_{ij} E_j \\ [H, P_i] = E_i & [H, K_i] = -P_i & [P_i, P_j] = -\varepsilon_{ij} B \\ [P_i, K_j] = -\delta_{ij} M & [P_i, J] = -\varepsilon_{ij} P_j & [K_i, K_j] = \varepsilon_{ij} \mathcal{K} \\ & [K_i, J] = -\varepsilon_{ij} K_j & \qquad i, j = 1, 2. \end{array} \tag{5.1}$$

It is interesting to point out that in a frame where the electric field $\vec{E}$ vanishes we recover the commutators corresponding to a pure magnetic Landau system [15, 26].

On the other hand, it is worthy to note that in order to relate the extended and nonextended GM algebras, we can redefine the basis generators inside the enveloping algebra taking into account the central character of M, $\mathcal{K}$ and B. So, we can write

$$K_i' = K_i + \lambda \varepsilon_{ij} P_j, \quad P_i' = P_i + \frac{M}{B} \varepsilon_{ij} E_j \tag{5.2}$$

where $\lambda = (-M + \sqrt{M^2 + \kappa B})/B$, and $M_e = \sqrt{M^2 + \kappa B}$ is a kind of effective mass [7]. Then, the new commutators entering K_i', P_i' are the same as above except that

$$[K_i', K_j'] = 0. \tag{5.3}$$

5.1 Coadjoint orbits

Let us denote by $(m, \kappa, \beta, \epsilon_1, \epsilon_2, h, p_1, p_2, k_1, k_2, j)$ the coordinates of an arbitrary point of $\mathcal{G}^*(2{+}1)$ in a basis dual to $(M,\, \mathcal{K},\, B,\, E_1,\, E_2,\, H,\, P_1,\, P_2,\, K_1,\, K_2,\, J)$. The coadjoint action

of an element $g=(\theta,\eta,c,\vec{d},b,\vec{a},v,\phi)\in GM(2{+}1)$ on the dual space $\mathcal{GM}^*(2{+}1)$ is given by

$$
\begin{aligned}
&m' = m,\\
&\vec{\kappa}' = \vec{\kappa},\\
&\vec{\beta}' = \vec{\beta},\\
&\vec{\epsilon}\,' = \vec{\epsilon}^{\,\phi} - \vec{v}\times\vec{\beta},\\
&h' = h - \vec{v}\cdot\vec{p}^{\,\phi} + \tfrac{1}{2}m\vec{v}^{\,2} - \vec{a}\cdot(\vec{\epsilon}^{\,\phi} - \vec{v}\times\vec{\beta}),\\
&\vec{p}\,' = \vec{p}^{\,\phi} - m\vec{v} - b\,(\vec{\epsilon}^{\,\phi} - \ \vec{v}\times\vec{\beta}) - \vec{\beta}\times\vec{a},\\
&\vec{k}\,' = \vec{k}^{\,\phi} + m\,\vec{a} + b\,\vec{p}^{\,\phi} - \tfrac{1}{2}b^2\vec{\epsilon}\,' + \vec{v}\times\vec{\kappa} - \vec{\beta}\times\vec{d} + \tfrac{1}{2}\vec{a}\times\vec{\beta},\\
&\vec{j}\,' = \vec{j} - \vec{a}\times\vec{p}^{\,\phi} - \vec{v}\times\vec{k}^{\,\phi} + \tfrac{1}{2}\,\vec{\kappa}\cdot\vec{v}^{\,2} + m\,\vec{a}\times\vec{v} + \tfrac{1}{2}\vec{\beta}\cdot\vec{a}^{\,2}\\
&\qquad\quad -\vec{d}\times\vec{\epsilon}^{\,\phi} + \tfrac{1}{2}b\,\vec{\epsilon}\,'\times\vec{a} - \vec{\beta}(\vec{d}\cdot\vec{v}).
\end{aligned}
\tag{5.4}
$$

Besides $m,\ \kappa$ and $C_0\equiv\beta$ we have the following invariants of the coadjoint action:

$$
C_1 = \vec{\beta}\,\vec{p}^{\,2} - 2\,m\,h\,\vec{\beta} + 2\,\vec{\beta}\,(\vec{\epsilon}\cdot\vec{k}) - 2\,\beta^2\,\vec{j} + \vec{\kappa}\,\vec{\epsilon}^{\,2}, \qquad C_2 = 2\,\vec{\beta}^{\,2}\,h + 2\,\vec{\beta}\cdot(\vec{p}\times\vec{\epsilon}) + m\,\vec{\epsilon}^{\,2}. \tag{5.5}
$$

These invariants are the nonrelativistic version of (4.3). The invariance of the magnetic field $\vec{\beta}$ may be seen as the consequence of the invariance of the Lorentz force $\vec{F}=\vec{\epsilon}+\vec{v}\times\vec{\beta}$ under (homogeneous) Galilei transformations. A charged particle moving slowly 'can see' mainly the magnetic field in our magnetic limit [25]:

$$
\frac{\sqrt{\vec{\epsilon}^{\,2}}}{\beta} \ll 1. \tag{5.6}
$$

Let us consider the relativistic invariant $C_0=\vec{\epsilon}^{\,2}-\beta^2$ from (4.3). Using (5.6) in the 0-term approximation term we obtain the nonrelativistic invariant

$$
C_0 = -\beta_0^2.
$$

We can omit the vector symbol because the magnetic field has only one component, so that

$$
\beta = \sqrt{\beta_0^2 + \vec{\epsilon}^{\,2}} \overset{(5.6)}{\approx} \beta_0 + \frac{1}{2}\frac{\vec{\epsilon}^{\,2}}{\beta_0}. \tag{5.7}
$$

By substituting

$$
h \to m(1+\frac{h}{m}), \qquad j \to -\kappa(1-\frac{j}{\kappa}), \qquad \beta \to \beta(1+\frac{1}{2}\frac{\vec{\epsilon}^{\,2}}{\beta^2})
$$

in the other invariants C_1 and C_2 of (4.3), and omitting terms of higher order in $h/m,\ j/\kappa$ and $\vec{\epsilon}^{\,2}/\beta^2$, we obtain their nonrelativistic counterparts of (5.5), respectively.

The classification of the coadjoint orbits is displayed in the Appendix. There are orbits of dimension 6 and 4, but the most important for us are the 6–D orbits denoted $\mathcal{O}^{C_1C_2}_{m\kappa\beta}$ with $\beta\neq 0$.

5.2 Symplectic structure

Each 6–D orbit $\mathcal{O}^{C_1C_2}_{m\kappa\beta}$ can be covered with one chart $(\mathcal{O}^{C_1C_2}_{m\kappa\beta}, \varphi)$. As coordinates we can choose $(\epsilon_1, \epsilon_2, p_1, p_2, k_1, k_2)$ since the Jacobian of the transformation

$$(m, \kappa, \beta, \epsilon_1, \epsilon_2, h, p_1, p_2, k_1, k_2, j) \to (m, \kappa, \beta, C_1, C_2, \epsilon_1, \epsilon_2, p_1, p_2, k_1, k_2)$$

equals $4\beta^4$ ($\neq 0$). The Poisson tensor Λ on the orbit $\mathcal{O}^{C_1C_2}_{m\kappa\beta}$ is

$$\Lambda = m\,\frac{\partial}{\partial k_i} \wedge \frac{\partial}{\partial p_i} + \kappa\,\frac{\partial}{\partial k_1} \wedge \frac{\partial}{\partial k_2} - \beta\,\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} - \beta\,\varepsilon^{ij}\,\frac{\partial}{\partial k_i} \wedge \frac{\partial}{\partial \epsilon_j}. \tag{5.8}$$

The components of Λ, written in matrix form, are

$$\Lambda^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta & -m & 0 \\ 0 & 0 & \beta & 0 & 0 & -m \\ 0 & -\beta & m & 0 & 0 & \kappa \\ \beta & 0 & 0 & m & -\kappa & 0 \end{pmatrix}, \tag{5.9}$$

The determinant of Λ is β^6 ($\neq 0$, for our orbits). The natural symplectic form on $\mathcal{O}^{C_1C_2}_{m\kappa\beta}$ is

$$\omega = \frac{\beta\kappa + m^2}{\beta^3} d\epsilon_1 \wedge d\epsilon_2 + \frac{m}{\beta^2} d\epsilon_i \wedge dp_i + \frac{1}{\beta}\varepsilon_{ij}\, dk_i \wedge d\epsilon_j + \frac{1}{\beta} dp_1 \wedge dp_2. \tag{5.10}$$

Therefore the coordinates $(\epsilon_i, p_i, k_i;\ i = 1, 2)$ are not canonical since the nonvanishing Poisson brackets are

$$\{\epsilon_i, k_j\} = -\varepsilon_{ij}\beta, \qquad \{p_1, p_2\} = -\beta, \qquad \{p_i, k_j\} = -\delta_{ij}\, m, \qquad \{k_1, k_2\} = \kappa. \tag{5.11}$$

Notice in particular that even the coordinates k_1, k_2 do not commute.

On the other hand, observe that the above tensor (5.9) coincides with (4.5) if we simply substitute h by m and j by $-\kappa$. The root of the proposed substitution is the fact that the Galilei–Maxwell group $GM(2{+}1)$ is the nonrelativistic limit of the Poincaré–Maxwell group $PM(2{+}1)$. We can look at κ as a nonrelativistic track of the angular momentum j (see also in this respect the arguments supplied in [4]).

Using the invariant C_2 from (5.5) we get a Hamiltonian linear in momenta $\vec{p}$ (which is the nonrelativistic version of (4.8)),

$$h = -\frac{\vec{p} \times \vec{\epsilon}}{\beta} - \frac{m}{2\beta^2}\vec{\epsilon}^{\,2} + \frac{C_2}{2\beta^2}. \tag{5.12}$$

Observe that by a naive limit $\beta \to 0$ we do not recuperate the free Hamiltonian (3.12); in fact $\lim_{\beta\to 0} h$ is not defined, and the same happens with the Poisson tensor.

After simple calculations we obtain the equations of motion

$$\dot{\epsilon}_i = 0, \qquad \dot{p}_i = -\epsilon_i, \qquad \dot{k}_i = p_i, \qquad i = 1, 2. \tag{5.13}$$

Of course, our system includes constant homogeneous fields $\vec{\epsilon}, \vec{\beta}$ perpendicular to each other as we already knew from the coadjoint action. We can also see that β does not take part in any of the formulas (5.13), so, surprisingly, the equations of motion are not affected by the magnetic field. Comparing formulas (5.13) with transformation rules (5.4) we conclude that these two sets of equations are compatible if we identify the parameter b with time. Moreover, the equations of motion are also independent of the parameter κ.

It would be interesting to compare our results with those obtained in a more standard way following the minimal coupling recipe. For simplicity we will assume here that the exotic extension κ vanishes. Let us consider a nonrelativistic particle with a unit negative electric charge moving on a plane in a constant homogeneous electric $\vec{\epsilon} = (\epsilon_1, \epsilon_2, 0)$ and magnetic $\vec{\beta} = (0, 0, \beta)$ fields. A phase space for this system is a 4–D symplectic manifold $(M, \tilde{\omega})$, where the differentiable manifold M is diffeomorphic to $\mathbb{R}^4$ with canonical coordinates denoted by π_1, π_2, r_1, r_2. The first pair $\vec{\pi} = (\pi_1, \pi_2, 0)$ is interpreted as the generalized momentum and the second $\vec{r} = (r_1, r_2, 0)$ is for the vector of position. The symplectic form is

$$\tilde{\omega} = d\pi_1 \wedge dr_1 + d\pi_2 \wedge dr_2. \tag{5.14}$$

The Poisson tensor $\tilde{\Lambda}$ in coordinates (π_1, π_2, r_1, r_2) takes the natural form

$$\tilde{\Lambda}^{ij} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

Finally, the Hamiltonian is represented by the minimal coupling expression

$$H = \frac{1}{2m}\left(\vec{\pi} + \frac{1}{2}\vec{\beta} \times \vec{r}\right)^2 + \vec{\epsilon} \cdot \vec{r}, \tag{5.16}$$

leading to the motion equations

$$\dot{\vec{\pi}} = -\vec{\epsilon} + \frac{1}{2}\vec{\beta} \times \dot{\vec{r}}, \qquad \dot{\vec{r}} = \frac{1}{m}\left(\vec{\pi} + \frac{1}{2}\vec{\beta} \times \vec{r}\right). \tag{5.17}$$

Integrals of the motion equations (5.17) are the Hamiltonian (5.16) and

$$C_1 = \vec{\epsilon} \times (\vec{\pi} - \tfrac{1}{2}\vec{\beta} \times \vec{r}\,), \tag{5.18}$$

$$C_2 = [\vec{\epsilon} \cdot (\vec{\pi} + \tfrac{1}{2}\vec{\beta} \times \vec{r})]^2 - 2\,(\vec{\epsilon} \cdot \vec{r})[\vec{\beta} \cdot (\vec{\pi} \times \vec{\epsilon}) - m\vec{\epsilon}^{\,2}]. \tag{5.19}$$

We can get canonical coordinates from the group coordinates $(\vec{p}, \vec{k})$, but unfortunately, there is not a point transformation connecting these two nonrelativistic interacting pictures.

6 Concluding remarks

The symmetry group of a system plus the formalism (Hamiltonian mechanics on a symplectic manifold) restrict the equations of motion, allow to define elementary systems, and may lead

to interacting systems compatible with the symmetries. The natural framework to display such symmetries is the method of coadjoint orbits. In this way we get a manifold, the invariant symplectic form, and the Hamiltonian.

This situation is quite different for the Hamilton formulation of mechanics in the phase space. In this frame the same symplectic manifold (M, ω) may be used to describe physical systems with different Hamiltonians. In order to build a Hamiltonian for interactions one is guided by other principles such as the minimal coupling rule. However, there is not a canonical way to display the symmetries in this context.

These two approaches have significative differences that could be appreciated along the examples worked in this paper. For instance, in our coadjoint orbit scheme, the Hamiltonians obtained for the interacting cases are linear in $\vec{p}$, the equations of motion depend on the electric field $\vec{\epsilon}$, while the magnetic field $\vec{\beta}$ takes part only in the symplectic two–form. These features are in sharp contrast to the usual interacting Hamiltonian in phase space.

Another difference is with respect to the role played by the fields. They are an integral part of the system in the group approach, while in the phase space they are treated as external parameters. The reason is that in our procedure we have considered the fields on the same foot as coordinates and momentum. In other words, the fields have been treated as dynamical fields instead of external fields, as usual. If we want to set a complete theory for the whole interacting system (particle + fields) it is expected that both components should take part of the system at the same level.

The group approach also enlighten us how to go from a relativistic to a nonrelativistic description of the system in a very simple and natural way. So, manifolds, symplectic forms, Hamiltonians, invariants and equations of motion are related through a contraction procedure.

The price for the simplicity of the group approach is the fact that we have to use noncanonical and noncommuting coordinates which obscure the physical interpretation. (However this is usual in the new formulations of planar physics, see for instance, [16, 27, 28, 29, 30]).

To show the explicit relation between the group approach and the formulation in phase space is an open problem. This situation of having different descriptions for the same system (one more appropriate to handle symmetries, the other adapted for an easier interpretation) happens also in quantum mechanics. In this framework symmetries can be described by unitary irreducible representation of symmetry groups in a representation space related to the coadjoint orbits as has been shown in Sections 3.3 and 4.3. On the other side, quantum mechanical systems are usually described by means of wavefunctions of the configuration space. The connection between these two pictures, sometimes is easy (the free case), but when interactions are included it is more involved.

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APPENDICES

A G-invariant Symplectic Structures

This Appendix contains some basic information about Poisson structures on a space $\mathcal{G}^*$ dual to a Lie algebra $\mathcal{G}$. We start from the definition of the Poisson bracket on some real n–dimensional manifold M, then we concentrate on the case when the manifold M is an orbit $\mathcal{O}^*$ of the coadjoint action. We prove that $\mathcal{O}^*$ is endowed in a canonical way with a symplectic structure.

Let M be a real n-dimensional differentiable manifold. The set of smooth real-valued functions $C^\infty(M)$ with a commutative multiplication constitutes a ring.

The Poisson bracket, $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$, on the manifold M is a bilinear relation satisfying the following conditions:

1. antisymmetry $\{f_1, f_2\} = -\{f_2, f_1\}$,
2. Jacobi's identity $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$,
3. derivation rule $\{f_1, f_2 f_3\} = f_2\{f_1, f_3\} + f_3\{f_1, f_2\}$,

for every $f_1, f_2, f_3 \in C^\infty(M)$.

The two first properties equip $\mathcal{C}^\infty(M)$ with the structure of a real Lie algebra. The derivation rule (known also as the Leibniz identity) and the bilinearity of the Poisson bracket say that for every $f \in \mathcal{C}^\infty(M)$ there exists a vector field X_f such that

$$X_f g = \{f, g\}, \qquad \forall g \in \mathcal{C}^\infty(M).$$

Let us cover some open subset $U \subset M$ by a chart (U, φ), such that $(x_1, \ldots, x_n)$ denotes the coordinates of $x \in U$ in this chart. In a natural basis $\frac{\partial}{\partial x_i} \equiv \partial_{x_i}$ $(i = 1, \ldots, n)$ induced by the chart we have $X_f = (X_f)^i\, \partial_{x_i}$, where Einstein's sum convention is used. It is easy to check that

$$(X_f)^i = X_f(x_i) = \{f, x_i\}. \tag{A.1}$$

Using (A.1) we find that the Poisson bracket

$$\{f, g\} = X_f g = (X_f)^i\, \partial_{x_i} g = \{f, x_i\}\frac{\partial g}{\partial x_i}. \tag{A.2}$$

On the other hand,

$$\{f, x_i\} = -\{x_i, f\} = -X_{x_i}(f) = -\{x_i, x_j\}\frac{\partial f}{\partial x_j} = \{x_j, x_i\}\frac{\partial f}{\partial x_j}. \tag{A.3}$$

Putting (A.3) into (A.2) we finally have

$$\{f, g\} = \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\{x_j, x_i\}. \tag{A.4}$$

Hence, it is enough to know the Poisson brackets of the coordinate functions $\{x_j, x_i\},\ i, j = 1, \ldots, n$, to compute the Poisson bracket of any pair of functions. The expression (A.4) defines a two-contravariant skew-symmetric tensor Λ by

$$\Lambda(df, dg) = \{f, g\}. \tag{A.5}$$

So, we conclude that the correspondence $f \to X_f$ defines a map $\pi(x) : T_x^*M \to T_xM$ in every point $x \in M$. The rank of $\pi(x)$ is called the rank of the Poisson structure in x.

The differential equations that determine the integral curves of X_f in M are

$$\frac{dx_i}{dt} = \{f, x_i\}, \qquad i = 1, \ldots, n. \tag{A.6}$$

They look like Hamilton equations being $-f$ the Hamiltonian function. For this reason the vector fields X_f are called Hamiltonian vector fields and f is called the Hamiltonian of X_f.

Let us consider now the problem of defining a Poisson structure on the space $\mathcal{G}^*$, dual to a Lie algebra $\mathcal{G}$. For every smooth function $f \in \mathcal{C}^\infty(\mathcal{G}^*)$ its (total) differential $(df)_x$ at $x \in \mathcal{G}^*$ is a linear mapping $(df)_x : T_x\mathcal{G}^* \to \mathbb{R}$, where $T_x\mathcal{G}^*$ denotes the tangent space of $\mathcal{G}^*$ at the point x. The dual space $\mathcal{G}^*$ is a vector space and it can be identified with $T_x\mathcal{G}^*$. The differential $(Df)_x$ is the functional over the tangent space $T_x\mathcal{G}^*$ and, hence, also over $\mathcal{G}^*$. It means that $(Df)_x \in (\mathcal{G}^*)^*$ which is isomorphic to $\mathcal{G}$. Thus, to any function $f \in \mathcal{C}^\infty(\mathcal{G}^*)$ we assign $\delta_x f \in \mathcal{G}$, in such a way that for every $y \in \mathcal{G}^*$

$$\langle y, \delta_x f\rangle = (d\,f)_x(y) = \frac{d}{dt} f(x + ty)|_{t=0}. \tag{A.7}$$

The formula (A.7) allows us to define the Poisson structure on $\mathcal{G}^*$ by

$$\{g, f\}(x) = \langle x, [\delta_x g, \delta_x f]\rangle. \tag{A.8}$$

Defining the functions $\xi_a \in \mathcal{C}^\infty(\mathcal{G}^*)$, with $a \in \mathcal{G}$, by $\xi_a(x) = \langle x, a\rangle$ we obtain from (A.7) that $\delta_x \xi_a = a$. Hence

$$\{\xi_a, \xi_b\}(x) = \langle x, [a, b]\rangle = \xi_{[a,b]}(x). \tag{A.9}$$

If $\{a_1, \ldots, a_n\}$ is a basis of $\mathcal{G}$ then $\xi_i \equiv \xi_{a_i},\ i = 1, \ldots, n$, can be chosen as a set of coordinate functions on $\mathcal{G}^*$. From (A.4) and (A.7) we obtain that the Poisson structure on $\mathcal{G}^*$ takes the form

$$\{f, g\}(x) = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\, c_{ij}^k \xi_k, \tag{A.10}$$

with c_{ij}^k being the structure constants of $\mathcal{G}$ relative to its basis $\{a_1, \ldots, a_n\}$. Thus, for a set of coordinate functions $\xi_i,\ i = 1, \ldots, n$, on $\mathcal{G}^*$ we have

$$\{\xi_i, \xi_j\} = c_{ij}^k \xi_k, \tag{A.11}$$

so that the Poisson bracket of $\{\xi_i, \xi_j\}$ is a linear function of ξ_k.

A special case of manifold with a Poisson structure is the symplectic manifold. A pair (M, ω) is called a symplectic manifold if M is a finite-dimensional differentiable manifold and

ω a nondegenerate 2-form satisfying the condition $d\omega = 0$. The nondegeneracy condition of ω is equivalent to the requirement that the rank of $\hat{\omega} : T_x(M) \to T_x^*(M)$, defined by

$$\langle \hat{\omega}(v), v' \rangle = \omega(v, v'),$$

be maximal at each point $x \in M$. The map $\hat{\omega}$ allows us to define a skewsymmetric contravariant tensor field Λ on M by

$$\Lambda(\alpha, \beta) = \omega(\hat{\omega}^{-1}(\alpha), \hat{\omega}^{-1}(\beta)), \qquad \alpha, \beta \in T^*M. \tag{A.12}$$

The tensor Λ constitutes the Poisson structure on M. Thus, when $\alpha = dg$ and $\beta = df$ we obtain that

$$\{g, f\} = \Lambda(dg, df) = \omega(\hat{\omega}^{-1}(dg), \hat{\omega}^{-1}(df)). \tag{A.13}$$

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. For every element $g \in G$ the inner automorphism $i_g : G \to G$ defined as

$$i_g(g') = g g' g^{-1}$$

induces a Lie algebra automorphism $i_{g^*} : \mathcal{G} \to \mathcal{G}$ which gives rise to the adjoint representation of G on $\mathcal{G}$ by $\mathrm{Ad}_g = \exp i_{g^*}$. The coadjoint representation of G on $\mathcal{G}^*$ is now given by

$$\langle \mathrm{CoAd}_g(u), a \rangle = \langle u, \mathrm{Ad}_{g^{-1}}(a) \rangle, \qquad u \in \mathcal{G}^*, \;\; a \in \mathcal{G}. \tag{A.14}$$

Each orbit $\mathcal{O}^*$ of the coadjoint action is a symplectic submanifold of the Poisson manifold $\mathcal{G}^*$, and it is endowed in a canonical way with a symplectic structure characterized by the two-form (Kirillov–Kostant-Souriau theorem)

$$\omega(X_a, X_b) = \langle u, [a, b] \rangle, \qquad u \in \mathcal{O}^*, \;\; a, b \in \mathcal{G}, \tag{A.15}$$

where X_a is the fundamental vector field associated with the coadjoint action

$$(X_a f)(u) = \frac{d}{dt} f\left(\mathrm{CoAd}_{e^{-ta}} u\right)|_{t=0} \, . \tag{A.16}$$

The Poisson structure on $\mathcal{O}^*$, as a submanifold of $\mathcal{G}^*$, defined through the symplectic 2-form ω (A.15) coincides with that induced by the Poisson structure (A.10) on $\mathcal{G}^*$.

B Coadjoint orbits classification for $\mathcal{GM}^*(2+1)$

Constraints	Dim	Invariants
$\beta \neq 0,\ m \neq 0,\ \kappa \neq 0$	6	$C_1 = \frac{1}{2}\vec{\varepsilon}^{\,2} \cdot \vec{\kappa} + (\vec{\varepsilon} \cdot \vec{k}) \cdot \vec{\beta} + \frac{1}{2}\vec{p}^{\,2} \cdot \vec{\beta} - mh\vec{\beta} - (\vec{j} \cdot \vec{\beta}) \cdot \vec{\beta}$ $C_2 = \vec{\beta}^2 h + (\vec{p} \times \vec{\varepsilon}\,) \cdot \vec{\beta} + \frac{m}{2}\vec{\varepsilon}^{\,2}$
$\beta = 0,\ m \neq 0,\ \kappa \neq 0$	6	$C_1 = -\frac{m}{2}\vec{p}^{\,2} + m^2 h - m\vec{\varepsilon} \cdot \vec{k} + (\vec{p} \times \vec{\varepsilon}\,) \cdot \vec{k}$ $C_2 = \vec{\varepsilon}^{\,2}$
$m = 0,\ \beta \neq 0,\ \kappa \neq 0$	6	$C_1 = \frac{1}{2}\vec{\varepsilon}^{\,2} \cdot \vec{\kappa} + (\vec{\varepsilon} \cdot \vec{k}) \cdot \vec{\beta} + \frac{1}{2}\vec{p}^{\,2} \cdot \vec{\beta} - (\vec{j} \cdot \vec{\beta}) \cdot \vec{\beta}$ $C_2 = h \cdot \vec{\beta} + (\vec{p} \times \vec{\varepsilon}\,)$
$\kappa = 0,\ \beta \neq 0,\ m \neq 0$	6	$C_1 = (\vec{\varepsilon} \cdot \vec{k}) \cdot \vec{\beta} + \frac{1}{2}\vec{p}^{\,2} \cdot \vec{\beta} - mh \cdot \vec{\beta} - (\vec{j} \cdot \vec{\beta}\,) \cdot \vec{\beta}$ $C_2 = \vec{\beta}^{\,2} h + (\vec{p} \times \vec{\varepsilon}\,) \cdot \vec{\beta} + \frac{m}{2}\vec{\varepsilon}^{\,2}$
$\beta = m = 0,\ \kappa \neq 0$	6	$C_1 = \vec{p} \times \vec{\varepsilon}$ $C_2 = \vec{\varepsilon}^{\,2}$
$\beta = \kappa = 0,\ m \neq 0$	6	$C_1 = -\frac{1}{2}\vec{p}^{\,2} + mh - \vec{\varepsilon} \cdot \vec{k}$ $C_2 = \vec{\varepsilon}^{\,2}$
$m = \kappa = 0,\ \beta \neq 0$	6	$C_1 = -\frac{1}{2}\vec{p}^{\,2} - \vec{\varepsilon} \cdot \vec{k} + \vec{\beta} \cdot \vec{j}$ $C_2 = h \cdot \vec{\beta} + \vec{p} \times \vec{\varepsilon}$
$\beta = m = \kappa = 0$	4	$C_1 = \frac{1}{2}\vec{p}^{\,2} + \vec{\varepsilon} \cdot \vec{k}$ $C_2 = \vec{p} \times \vec{\varepsilon}$ $C_3 = \vec{\varepsilon}^{\,2}$ $C_4 = (\vec{p} \cdot \vec{\varepsilon}\,) \cdot (\vec{p} \times \vec{\varepsilon}\,) + \vec{\varepsilon}^{\,2} \cdot (\vec{k} \times \vec{\varepsilon}\,)$